

Inverse scattering: applications to nuclear physics

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Review commissioned for Scholarpedia

Draft of March 3, 2013

Abstract: In what follows we first set the context for inverse scattering in nuclear physics with a brief account of inverse problems in general. We then turn to inverse scattering which involves the S-matrix, which connects the interaction potential between two scattering particles with the measured scattering cross section. The term ‘inverse’ is a reference to the fact that instead of determining the scattering S-matrix from the interaction potential between the scattering particles, we do the inverse. That is to say, we calculate the interaction potential from the S-matrix. This review explains how this can now be done reliably, but the emphasis will be upon reasons why one should wish to do this, with an account of some of the ways this can lead to understanding concerning nuclear interactions.

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1 General introduction: what are inverse problems?

The subject of this review is a very specific part of the research field ‘Inverse problems’, a field of vast scope, with dedicated journals including: Inverse Problems in Science and Engineering (ISSN 1741-5977, print, and 1741-5985, online), and, Inverse Problems (ISSN 0266-5611, print, and 1361-6420, online).

Inverse problems can be understood by contrast with the corresponding direct problems, as some examples should make clear:

1. Given a distribution of electrical current within the brain, it is a straightforward *direct* problem to calculate the magnetic fields outside the head; the *inverse* problem of calculating the currents from measured magnetic fields is the much more difficult ‘biomagnetic inverse problem’.
2. Determining the size and location of a mass of iron ore from sensitive variations of gravity at the Earth’s surface is much harder than the straightforward direct problem of calculating fluctuations in gravity at the surface due to known mass concentrations.
3. Various forms of tomography that are central to modern medicine can be seen as inverse problems.

Such problems have given rise to an active subdiscipline of applied mathematics centred around integral equations; browsing recent issues of the journals mentioned above will give a flavour of the subject, and hint at its importance in modern pure and applied science.

The subject of this article is much more restricted: the application of inverse scattering in nuclear physics. The nuclear inverse problem shares one key property with all of those mentioned here: it is much harder and generally less well-developed than the corresponding direct problem. It differs somewhat in that each of the other inverse problems is widely accepted and plays a key role in the relevant scientific, medical or commercial activity. The usefulness of inverse methods in nuclear physics is less well-known; this article will give examples of where it has been useful, and maybe will inspire some new applications that this author has not considered.

2 Introduction to inverse scattering in nuclear physics

In this article, inverse scattering chiefly refers to *the determination of a local scattering potential that yields a given set of S-matrix elements*. Although ‘true’ nuclear interactions are understood to be non-local, we discuss only the derivation of local representations of the S-matrix. There always exists such a representation and in view of the wide range of possible forms of non-locality, the determination of a non-local potential from a single set of S-matrix elements at a single energy will be under-determined

and we do not discuss this. We do remark, however, that inversion can be the most natural way of determining a local equivalent (in the sense of yielding the same S-matrix) of a non-local potential. Inverse scattering can also be extended to include the determination of the interaction potential directly from scattering observables (from this viewpoint, optical model fitting is an elementary form of inverse scattering.) The *inverse* in ‘inverse scattering’ indicates a contrast with the (much easier) *direct* scattering problem in which the S-matrix, and thus the scattering observables, are calculated from an interaction potential. The physical context in which inverse and direct scattering are discussed in this article is: the scattering of one microscopic body from another in a model where the interaction between the bodies is described by a potential and the natural solution involves using this potential in the Schrödinger equation. In accord with the title of this article, the microscopic bodies that predominantly feature in this article are pairs of atomic nuclei, with one of them often a nucleon.

This review sets out to do the following:

1. Define the categories of nuclear inverse scattering cases, specifying those that will be given fuller treatment in this review and giving references for those that will not.
2. Present an overview of the various methods that have been applied to the nuclear inverse scattering problem.
3. Present an account of a particular inversion procedure, the iterative-perturbative (IP) inverse scattering algorithm, that has had a wide range of applications. This will include specific examples of what it can do.
4. Show ‘what inversion can do for you’. This takes the form of a range of examples showing what kind of information or understanding concerning nucleus-nucleus interactions can be obtained using inversion. At the end, we leave it up to the imagination of the reader to extend that range.

This review is specifically *not* a comprehensive review of the formal theory of inverse scattering.

3 Definitions and notation

The *direct* scattering problem, which is in the background to this review, is the calculation of the scattering of interacting microscopic particles, typically but not exclusively a nucleon and an atomic nucleus. We assume that there is an interaction potential between the interacting particles that can be substituted into the time independent Schrödinger’s equation¹. This can be solved for the radial wave function for specific

¹We do not consider wave packets nor do we consider the justification for the stationary state treatment of scattering.

values of the orbital angular momentum l (we generalize shortly to the case where the particle has spin) having asymptotic form

$$u_l(r) \rightarrow I_l(r) - S_l O_l(r). \quad (1)$$

Eqn. 1 defines the S-matrix S_l for orbital angular momentum quantum number l for a spinless projectile; $I_l(r)$ and $O_l(r)$ are the ingoing and outgoing radial solutions, Coulomb wave functions, for the case where the projectile and target are both charged. The S-matrix is frequently expressed in terms of phase-shifts δ_l : $S_l = \exp 2i\delta_l$. For complex potentials, δ_l is complex and $|S_l| \leq 1$ to preserve unitarity.

For a spin- $\frac{1}{2}$ projectile, we have

$$u_{lj}(r) \rightarrow I_l(r) - S_{lj} O_l(r) \quad (2)$$

where $j = l \pm \frac{1}{2}$. The case of spin-1 projectiles, such as deuterons, can also be handled; this involves the inversion of a coupled channel S-matrix:

$$u_{l'l}^J(r) \rightarrow \delta_{l'l} I_{l'}(r) - S_{l'l}^J O_{l'}(r) \quad (3)$$

where J is the total angular momentum (assuming spin zero target nucleus), $l, l' = J \pm 1$ and have the same parity; for $l = l' = J$ there is no coupling.

The *inverse* problem in these cases reverses the situation: given S_l , S_{lj} or $S_{l'l}^J$, what is the interaction potential? We shall also touch on the inverse problem of establishing S_l , S_{lj} or $S_{l'l}^J$ from measured observables.

General references: The book by K. Chadan and P.C. Sabatier [1] presents a comprehensive account of inverse scattering with an emphasis on the formal aspects. We refer to this as CS89. An old but still useful introduction to inversion is Chapter 20 of the book by R.G. Newton [2].

Review article: The review, ‘The application of inversion to nuclear physics’, by Kukulin and Mackintosh [3] reviews inverse scattering particularly as applied to nuclear scattering, up to 2003 and provides a much more comprehensive bibliography than the present one. We refer to it as KM04.

Conference proceedings: Many articles discussing the theory and application of inverse scattering, as well as broader aspects of inversion, can be found in the conference proceedings: [4, 5, 6].

4 Categories of inverse scattering

In this review we shall refer to the following categories of inverse scattering:

1. Fixed- l inversion

Given $S_l(E)$ for all energies E at a fixed l , determine the potential $V(r)$ that gives S_l . This is the classical inversion problem solved by Gel’fand and Levitan and also

Marchenko, see CS89. The term ‘fixed- l inversion’ is misleading since it has been generalized to include derivation of spin-orbit and tensor terms for specific fixed J and parity, see Sections 6.1 and 7.1.

2. Fixed energy inversion

Given S_l for all l at a specific energy E , fixed E inversion determines $V(r)$ that reproduces S_l at energy E . We include under this heading inversion $S_{lj} \rightarrow V(r) + \mathbf{l} \cdot \mathbf{s} V_{\text{SO}}(r)$ and also the generalization for spin-1 projectiles leading to the determination of a specific tensor interaction from $S_{ll'}^J$ (where $(-1)^l = (-1)^{l'}$). Scattering from target nuclei with spin can sometimes be treated by determining independent interactions for different values of the channel spin. Applications in which more general non-diagonal S-matrices are inverted to non-diagonal potentials (coupled channel inversion) have been discussed.

3. Mixed case inversion

At low energies, there may be too few active partial waves for satisfactory inversion. If there exist sets of S_l at closely spaced energies, they can be inverted together to determine a potential in ‘mixed case’ inversion, having aspects of fixed- l and fixed- E inversion. It can be viewed as incorporating information from the local energy dependence of S_l . Mixed case inversion is possible with the iterative-perturbative (IP) inversion procedure that is presented in Section 6.2.3.

4. Energy-dependent inversion

Nuclear potentials are inherently energy dependent. Given S_l for a wider range of energies than is appropriate for mixed case inversion, energy-dependent inversion determines a potential with an appropriately parameterized energy dependence, and can be considered a generalization of mixed case inversion.

5. Variants of fixed energy, mixed case and energy-dependent inversion

Scattering of identical bosons provides S_l for just the even values of l . Where there are sufficient active partial waves, this situation can be handled straightforwardly by the IP method and semi-classical (WKB) methods. Likewise, it is often straightforward to obtain with IP inversion a parity-dependent potential in cases where exchange processes (for example) require separate potentials for odd-parity and even-parity partial waves (see Remark 4, below).

6. Direct observable-to-potential inversion

Using IP inversion, it is possible to combine in one algorithmic procedure $S_l \rightarrow V(r)$ inversion together with a determination of S_l from a fit to data. This can be applied with any of types 2 to 5 above. (This is distinct from the two-step inversion mentioned in Remark 2 below.)

Remark 1. In principle, type 1 (fixed l) requires S_l for all energies and type 2 (fixed E) requires S_l for all l , but practical implementations have been developed. For fixed energy inversion, this allows a potential to be defined out to a specific radius to be determined from a set S_l over an appropriate range of l . This effectively puts a lower limit to the energy for which a potential can be determined.

Remark 2. In principle all methods are subject to problems of non-uniqueness and errors, though these problems can often be minimized in practice. Important for this is the fact that practical inversion methods may allow the inclusion of *a priori* information, especially in cases tending to be under-determined.

Remark 3. In practical applications, the S-matrix to be inverted generally comes from theory or from fits such as R-matrix fits or effective range fits to measured observables over a range of energies. There have also been applications in which S_l have been determined by fitting observables at a single energy with a direct search. Such S-matrix fitting is also an inverse problem and the technical and formal aspects of observable-to- S_l inversion are to be found elsewhere, e.g. CS89. We do mention some applications of the resulting ‘two-step’ phenomenology, which might be considered as an alternative form of model-independent optical model (OM) fitting, having certain advantages. This is true also of type 6, direct observable-to-potential inversion.

Remark 4. Methods for both fixed- l and fixed- E exist for including the energy of bound states as input information for the inversion.

Remark 5. A parity-dependent component (e.g. real or imaginary central, real or imaginary spin-orbit) of a potential may be written $V_W + (-1)^l V_M$ and, in the context of parity dependence, we refer in what follows to V_W and V_M as the Wigner and Majorana components. We reserve the term ‘ l -dependent’ for other forms of partial wave dependence, never for parity dependence.

5 Alternative inversions

As an alternative to determining the potential that reproduces the S-matrix, one can determine the potential that reproduces the radial wave function for a given partial wave. The trivially equivalent local potential (TELP) of Franey and Ellis [7] is necessarily l -dependent. However l -weighted TELPs, applicable for all l , can be constructed (as in the CC code FRESKO [8]) and are widely used to represent dynamic polarization potentials, DPPs, (discussed below). As suggested by, for example, the somewhat arbitrary nature of the partial wave weighting, weighted TELPs cannot be expected to give the same potential as $S_l \rightarrow V(r)$ inversion and actual comparison [9] confirms that indeed there are substantial differences. The consideration of such alternative ways of defining a local potential can throw light on the physics of local potential models of scattering, as discussed by M. S. Hussein, *et al* [10].

As an alternative to the l -dependent TELP, one can produce a spatial representation of the potential $V(\mathbf{r})$ that reproduces the elastic channel wave function $\psi(\mathbf{r})$ throughout the nucleus; note the vectorial dependence upon \mathbf{r} . Such a representation illuminates the non-locality induced by channel coupling, showing regions where flux leaves and then returns to the elastic channel. The real and imaginary parts of the ‘ ψ -potentials’ are determined, respectively, from the real and imaginary parts of $\psi(\mathbf{r})^* \nabla^2 \psi(\mathbf{r})$ as described in Ref. [11], and for spin- $\frac{1}{2}$ projectiles, Ref. [12] and references therein. Ref. [13] compares the very different wave functions within the nucleus for l -dependent and l -

independent potentials that have the same asymptotic form, i.e. the same S_l .

6 Methods for $S_l \rightarrow V(r)$ inversion

A number of techniques for $S_l \rightarrow V(r)$ inversion have been put forward and here we list the most significant with the emphasis on the historically significant and those that have been widely applied, leading to the understanding of nuclear interactions.

In principle, as has been pointed out by Chadani and Sabatier [1], the nuclear scattering inverse problem is under-determined and hence subject to ambiguities. This is more of a problem for formal methods that do not readily permit the inclusion of prior information. In fact, it proves not to be a problem in many practical cases, especially where the sought-for potential is not too far from some known potential, as, for example, when determining a dynamic polarization potential. In other cases, it does matter, and the inclusion of prior information in the overall inversion problem has to be accepted as reasonable. It's a strength of certain inversion procedures that this is possible with them. A specific example will be given.

A possible consequence of under-determination is the occurrence of rapid wiggles on the potentials that are determined. In effect, potentials $V(r)$ and $V(r) + \tilde{V}(r)$, where $\tilde{V}(r)$, a 'null potential' [14], is a function in the form of a set of short wavelength oscillations, have exactly, or very nearly the same S_l . This is particularly significant since genuine wavy features do occur, e.g. as a consequence of underlying l -dependence. Such genuine waviness must be distinguished from the spurious. The IP method does afford means for such discrimination.

6.1 Fixed- l methods

The inversion formalism of Gel'fand, Levitan and Marchenko [1] can be made to yield a spin-orbit potential from S_{lj} and also (in coupled-channel form) yield a tensor force when different l values contribute to the S-matrix for specific total angular momentum J and parity π . The problems are: (i) nuclear potentials are typically energy dependent but the method relates S_l to a single potential for the whole energy range, and (ii) a very wide energy range is required to determine the potential. Even for nucleon-nucleon scattering, where the method has been applied, the pion threshold limits the energy range.

6.2 Fixed- E methods and extensions

Newton [2, 15], starting from the fixed- l formalism of Gel'fand and Levitan, devised a restricted fixed- E inversion procedure that was further developed by Sabatier [1] and others (see Section 6.2.1) into the Newton-Sabatier (NS) inversion method. This for-

malism directly derives a potential from S-matrix elements and is formally exact. The related method of Lipperheide and Fiedeldey [16, 17] starts from a specific parameterization of S_l .

Fixed- E inversion methods for application at higher energies based on the JWKB approximation and other semi-classical approximations have been developed, see Kujawski [18] and others [19]. For applications see Section 6.2.2.

The most widely applied inversion method is the iterative-perturbative (IP) algorithm [20] based upon the generally linear response of S_l to changes in $V(r)$ [21]. IP inversion has been extended to handle mixed-case inversion, energy-dependent inversion, spin- $\frac{1}{2}$ inversion and some cases of spin-1 inversion leading to a tensor interaction.

A number of other approaches to fixed- E inversion are referenced in the review [3]. The subject of fixed- E inversion is a topic of on-going research, see e.g. [22].

6.2.1 Newton-Sabatier and related methods

The formal Newton-Sabatier (NS) inversion method was developed into a practical applicable method in the important work of Münchow and Scheid [23] (MS). Key aspects were the matching of the radial range to the range of l -values for which S_l was provided, and the adoption of an over-determined matrix algorithm.

For the later extension of the MS method to spin- $\frac{1}{2}$ see Ref. [3].

Formal inversion methods of this kind simply translate a set of S_l values to a function $V(r)$ with the disadvantage that when, for example, suspected ‘noise’ in the input S_l leads to oscillatory features in $V(r)$, it is not possible to adjust the precision required of the inversion to evaluate the physicality of these features. It is also not straightforward to include prior information concerning the potential; such information is useful in difficult applications and for eliminating the effects of the general under-determination of nuclear inverse scattering [1]. Test inversions do appear to exhibit some spurious oscillations that might be difficult to distinguish from genuine waviness, see above,

It seems that there have been rather few papers exploiting the MS-NS method to extract information about nuclear scattering. However, the formal developments by Newton and his successors have been of great value, not least for showing that there always is a local potential corresponding to an appropriate radial range for a corresponding range of partial waves.

An inversion procedure of a similar kind, that due to Lipperheide and Fiedeldey [16, 17], LF, was actually the first to extract information concerning nuclear interactions by means of inversion: the long range interaction generated by Coulomb excitation of heavy ions, see [24]. With MS-NS or LF $S_l \rightarrow V(r)$ inversion, $V(r)$ is uniquely determined by S_l , so prior information must sometimes be included in the determination of S_l . An application [25] to the analysis of scattering data illustrates this. The resulting potential is compared by Brandan and Satchler [26] and adjudged less physical than

an alternative described in Section 7.2.1. We emphasize that this is not a criticism of LF $S_l \rightarrow V(r)$ inversion except insofar as the the inversion procedure [16, 17] requires that S_l must be represented rather precisely in a specific multi-term rational function form.

6.2.2 Semiclassical inversion, WKB methods

WKB methods [18, 19] are expected to work well at higher energies. The implementation of these methods is described in the following papers in which inversions exploiting the WKB approximations have been carried out, Refs. [27, 28]. The WKB inversion procedure has also been exploited in an interesting study of the effects of systematic errors in the analysis of nuclear scattering data [29].

6.2.3 Iterative-perturbative (IP) inversion

IP inversion [20, 30] exploits the relatively linear response [14, 21] of S_l to changes in $V(r)$ to construct a procedure based on the iterative correction of a ‘starting reference potential’, SRP. The SRP in many cases can be a zero potential. Ref. [20] demonstrated the method in a calculation of the dynamic polarization potential for the breakup of ${}^6\text{Li}$. IP inversion was independently developed by Kukulin [31]. The extension to spin- $\frac{1}{2}$ was presented in Ref. [32], the introduction of error analysis in Ref. [33] and mixed case inversion in Ref. [34]. More details of IP inversion and its extensions are given in Section 6.3, and applications will be described in Section 7. The extension of IP inversion to data-to-potential direct inversion is described in Section 7.3.

6.2.4 Other fixed- E inversion methods

There are various other inversion techniques referred to in [3] and [4, 5, 6]. These sources cite many references that are valuable for understanding the formal issues connected with inversion, but few of the other techniques seem to have yielded information concerning nuclear interactions.

6.3 The IP method and its extensions

A key feature of IP inversion is that it is not tied to specific analytic properties of Schrödinger’s equation, but simply to the fact that the response of the S -matrix to changes in the potential is approximately linear. This near linearity leads to properties which give IP inversion a powerful advantage as a practical tool. These include:

- (i) It is highly generalizable. Hence, for example, inclusion of spin-orbit and even tensor forces requiring coupled channel extensions, are relatively straightforward and do not compromise the accuracy of the inversion.
- (ii) Mixed-case and energy-dependent inversion, as defined above, are possible.

- (iii) Useful information can be obtained when the input data are noisy and incomplete. The iterative procedure can be halted before S -matrix elements are inverted to a greater precision than is warranted by their own precision.
- (iv) IP inversion can be incorporated into a one-step observable-to-potential inversion algorithm.

The IP approach described here determines a *local* potential corresponding to given S -matrices as calculated by Schrödinger's equation; however it has also been applied [36] to determine a Dirac potential simply by applying the appropriate transformation to the extracted Schrödinger potential.

IP inversion is implemented in the Fortran 90 code Imago. An Imago users's manual [37], and also Ref. [3], give a more general and detailed account of the formalism than is given below, which presents the basic idea. An earlier short review of IP inversion and its applications, including some examples not discussed here, can be found in Ref. [35].

6.3.1 Basic IP method

The IP approach is based on the fact that, in general, the scattering matrix (or phase shift) responds in a remarkably linear way to changes in the scattering potential; explicit examples are given in the appendix of Ref. [38]. This makes possible a step-wise linearization procedure in which the potential corresponding to some given S -matrix can be established in a series of iterations starting from a guessed potential.

The linear response of the S -matrix to changes in V , which lies at the heart of the IP method, can be expressed in various equivalent forms. The change, δS_l , in the scattering matrix $S_l(k) = \exp 2i\delta_l(k)$ for partial wave l and CM energy $E = \hbar^2 k^2 / 2m$ induced by a small change² in the scattering potential, $V(r) \rightarrow V(r) + \Delta V(r)$, is

$$\delta S_l = i \frac{m}{\hbar^2 k} \int_0^\infty \psi_l^2(r) \Delta V(r) dr \quad (4)$$

where ψ_l is a regular solution of Schrödinger's equation with the asymptotic normalization:

$$\psi_l(k, r) \rightarrow I_l(r) - S_l(k) O_l(r) \quad (5)$$

and I_l and O_l are the conventional incoming and outgoing Coulomb wave functions. This well known result follows immediately from the Wronskian relationships for I_l and O_l . These functions can be written in terms of the regular and irregular Coulomb solutions $F_l = i(I_l - O_l)/2$ and $G_l = (I_l + O_l)/2$. If the Coulomb interaction is absent, F_l and G_l become spherical Bessel j_l and Neumann h_l functions: $F_l \rightarrow j_l(kr)$ and $G_l \rightarrow h_l(kr)$.

If the target S -matrix to be inverted is S_l^{tar} , then the aim is to determine the potential

²When spin-orbit interactions are considered, we add relevant labels, e.g. S_{lj} .

for which the S-matrix S_l^{inv} renders the sum:

$$\sigma^2 = \sum_l^{N_l} |S_l^{\text{tar}} - S_l^{\text{inv}}|^2 \quad (6)$$

as close to zero as possible, or at least as close as is reasonable. We add that qualifier since the target S-matrix will have numerical imprecision or errors or superimposed ‘noise’; it is a virtue of the IP method that precise inversion can be avoided where it is inappropriate. Eqn. 6 omits any labels relating to spin. We shall also omit labels on components of the potentials (real, imaginary, real and imaginary spin-orbit etc) which are included in Refs. [37] and [3].

The minimization is carried out iteratively, starting from V_{SRP} , the *starting reference potential* (SRP). In favourable cases, the SRP can be a zero potential. An iteration is carried out as follows: if previous iterations lead to the current potential $V(r)^{\text{curr}}$, the next step is to find

$$V(r)^{\text{new}} = V(r)^{\text{curr}} + \sum_{i=1}^{N_b} \lambda_i v_i(r) \quad (7)$$

where the λ_i are amplitudes to be determined and the $v_i(r)$ are members of the *inversion basis* of dimension N_b . For the inversion basis, the inversion code Imago [37] offers a choice including: displaced gaussian functions, zeroth order Bessel functions, spline functions and others. To determine amplitudes λ_i such that the current S-matrix S_l^{curr} may become the target S-matrix S_l^{tar} (or, at least, closer to S_l^{tar}), we identify $\Delta_l = S_l^{\text{tar}} - S_l^{\text{curr}}$ with the change δS_l expressed in Eqn. 4 resulting from the perturbation $\Delta V = \sum \lambda_i v_i(r)$, to get

$$\Delta_l = \sum_i^{N_b} \lambda_i \frac{im}{\hbar^2 k} \int_0^\infty \psi_l^2(r) v_i(r) dr \equiv \sum_i^{N_b} M_{li} \lambda_i \quad (8)$$

where $\psi_l(r)$ is the regular solution involving the current potential $V(r)^{\text{curr}}$. The next step involves matrix algebra to determine the best set λ_i for the (in general) over-determined system Eqn. 8; note that $N_b \leq N_l$, the number of l values included in the inversion. In the original IP work [20], and in some subsequent work, we followed Ref. [23] in their use of the standard matrix method. Using natural matrix notation, Eqn. 8 can be written $\mathbf{\Delta} = \mathbf{M}\mathbf{\lambda}$ leading to:

$$\mathbf{\lambda} = (\mathbf{M}^\dagger \mathbf{M})^{-1} \mathbf{M}^\dagger \mathbf{\Delta}, \quad (9)$$

with \mathbf{M}^\dagger the hermitian adjoint of \mathbf{M} . From these values of λ_i , the new current potential can be calculated from Eqn. 7. Further iterations can be carried out until a suitable low level of σ is reached.

Before describing the practical implementation, which involves sequences of iterations rather than a single sequence, we note that an alternative to the direct inversion of Eqn. 9 has proven superior: singular value decomposition, SVD, see, e.g. Ref. [39]. SVD makes convergent iteration possible in cases where the direct matrix method fails. The first step is to re-write \mathbf{M} in the following product form:

$$\mathbf{M} = \mathbf{U} \mathbf{D} \mathbf{V}^\dagger \quad (10)$$

where \mathbf{V} is square, $\mathbf{V}\mathbf{V}^\dagger = 1$ and $\mathbf{U}\mathbf{U}^\dagger = 1$. Matrix \mathbf{U} will not be square since we are, in general, dealing with an over-determined system. Matrix \mathbf{D} is diagonal with elements d_j for $j = 1, \dots, N_b$. We can then write

$$\boldsymbol{\lambda} = \mathbf{V}\mathbf{D}^{-1}\mathbf{U}^\dagger\boldsymbol{\Delta} \quad (11)$$

where \mathbf{D}^{-1} is diagonal with elements $1/d_j$. In general, d_j vary over many orders of magnitude. The smallest d_j are the least accurately determined, and so can be eliminated. The program Imago does this by setting a tolerance limit, with any elements d_j that are below that limit being set to zero. This limit can be lowered in successive sequences of iterations as we now describe.

The iterative inversion is not carried out in a single sequence, but in a series of discrete sequences of iterations. This allows divergences or oscillatory potentials, following too large an inversion basis or too small an SVD tolerance, to be avoided. A sequence of iterations leading to a modest reduction in σ without divergence or spurious oscillations can be followed by another sequence that has a basis with a larger N_b or wider radial range, and/or a smaller SVD tolerance. This will then, typically, converge to a lower value of σ . After any sequence of iterations, it is possible to backtrack if the chosen inversion parameters lead to divergence or to a potential with oscillatory features that might be spurious. In practice, depending on the case, one or several sequences of iterations will lead to a potential that gives a very close fit to the S-matrix without spurious oscillations. As implemented in Imago, the fit to both the S-matrix and the observables can be seen interactively, on-screen, after each sequence of iterations. Interestingly, σ can often be reduced by an order of magnitude by further iterations even after a visually nearly perfect fit to S_l has been achieved over almost the entire l -range over which $1 - |S_l|$ is appreciable. Furthermore, for cases with many partial waves, a perfect fit to the observables at far backward angles requires an exceptionally low σ , sometimes much lower than required for a good visual fit to S_l .

It is a matter of good practice to test the uniqueness of the inversion by verifying that the same result is obtained with a different SRP or inversion basis.

Following a converged inversion procedure, one can be assured that a potential has indeed been found that reproduces the input S-matrix to a precision quantified by σ and verified by visual fits to both the S-matrix and the observables (as mentioned, the latter often being more sensitive). In addition, the degree of uniqueness can be tested in the way just described. In the case of input S_l that is noisy or less well determined, the iterative process can be stopped at a larger value of σ and useful information extracted, perhaps from a range of alternative solutions. This can happen in cases of heavy ion scattering at low energies where there is little useful information in S_l for low values of l where $|S_l|$ is typically very small.

6.3.2 Spin- $\frac{1}{2}$ inversion

The generalization of IP inversion to the spin- $\frac{1}{2}$ case [40], $S_{lj} \rightarrow V(r) + \mathbf{l} \cdot \mathbf{s} V_{\text{so}}(r)$, is straightforward and its implementation in the code Imago is described in Ref. [37]. For

spin- $\frac{1}{2}$ scattering the potential becomes,

$$V_{\text{cen}}(r) + iW_{\text{cen}}(r) + 2\mathbf{l} \cdot \mathbf{s} (V_{\text{so}}(r) + iW_{\text{so}}(r)). \quad (12)$$

With suitable indexing, it is straightforward to expand the matrix system to accommodate the new terms. An independent inversion basis for the spin-orbit term can be defined and may differ from that for the central term in two respects: (i) it does not extend quite to the origin, since only the $l = 0$ radial wave function is non-zero there and $\mathbf{l} \cdot \mathbf{s}$ is zero for $l = 0$, and, (ii) it need not extend to such a great radius, since, in fact, spin-orbit terms tend to be small at a radius where central terms are still finite.

6.3.3 Mixed case and energy-dependent inversion

Mixed case inversion determines the potential that reproduces $S_{ls}(E_i)$ for a few discrete energies E_i . This is particularly useful at low energies and with light target nuclei when only a few partial waves are active — too few to define the potential for a single energy. It is quite straightforward to expand the matrix system to include multiple energies and multiple sets of target S_{ij} . Because the nuclear optical potential is intrinsically energy dependent, mixed case inversion is useful only over relatively narrow ranges of energy, unlike energy dependent inversion.

Energy dependent IP inversion allows the parameters of postulated energy dependent functions to be optimized in an expansion of the matrix system described above. Ref. [41] describes the formulation as originally applied and as is implemented in Imago. A significant feature is that different parameterized forms must be used for the real and imaginary components. This is obviously important in the case discussed in Ref. [41] where the energy range involved crossed the inelastic threshold for $p + {}^4\text{He}$ scattering.

6.3.4 Parity dependence and identical bosons

Parity-dependent inversion determines, in effect, separate interactions for the even-parity and odd-parity partial waves. Parity-dependent terms can also be straightforwardly generated by including a Majorana inversion basis $v_i^{\text{M}}(r)$ so that any term x , where x could refer to real central, imaginary central, real spin-orbit or imaginary spin-orbit, can have an added Majorana term of the form $(-1)^l \sum_i \lambda_i^{\text{M}} v_i^{\text{M}}(r)$.

Inversion for the scattering of identical bosons involves inversion for which S_l exists only for even l . The only issue for IP inversion is whether there are enough partial waves to define the potential with sufficient accuracy. Section 7.2.1 describes several cases of inversion involving identical bosons at fairly high energies for which there were plenty of partial waves for satisfactory inversion. For low energy identical bosons involving S_l calculated from theory, it might be possible to interpolate to odd l .

6.3.5 Including bound state energies

It is possible to include the energies of bound states as input to the inversion process. This will be useful at low energies where, even with an S-matrix for multiple energies in energy dependent inversion, there is a paucity of information with which to define the potential. The method was introduced by Cooper and fully described in [81]. In that reference, a fit to the energy of an $l = 1$ bound state in ${}^7\text{Be}$ was included in an inversion of low energy S_{lj} for ${}^3\text{He}$ scattering on ${}^4\text{He}$, leading to a parity-dependent potential.

6.3.6 Spin-1 inversion yielding tensor interaction

The extension of the IP algorithm to spin-1 inversion is not as straightforward as the extensions described above. This follows from the fact that the scattering of spin-1 projectiles from a spin-zero target requires, in general, a coupled channel calculation. For example for total angular momentum $j = 1$ and positive parity, the $l = 0$ and $l = 2$ channels may be coupled by a tensor force interaction. The general S-matrix for spin-1 projectiles scattering from a spin-zero target may be written $S_{ll'}^j$, where $l = j - 1, j$ and $j + 1$. Channels with $l = j - 1$ and $j + 1$ will, in general, be coupled. The possible tensor interactions were classified by Satchler [42, 43] and labeled T_R , T_P and T_L . The third of these is believed to be small, and the T_P interaction appears to be hard to distinguish phenomenologically from the first, T_R , interaction. The T_P interaction might be important, but involves gradient operators for which an inversion procedure has not been devised. We therefore assume that the inter-nuclear interaction may contain a tensor force component of the form:

$$T_R V_R(r) \equiv ((\mathbf{s} \cdot \hat{\mathbf{r}})^2 - 2/3) V_R(r). \quad (13)$$

To invert an S-matrix of the form $S_{ll'}^j$ to determine a potential including a T_R interaction, requires coupled channel inversion in which a non-diagonal S-matrix yields a non-diagonal potential. A specification of the formalism together with an account of tests of its performance can be found in Ref. [44]. This reference also presents the results of inverting $S_{ll'}^j$ generated with a T_P interaction leading to a T_R interaction. The tests in Ref. [44] were for a single energy case with no parity dependence, but did include an extension to direct data-to-potential inversion, see Section 7.3.

Spin-1 inversion leading to a T_R interaction is straightforwardly extended to include parity dependence, Section 6.3.4 in all components as well as energy dependence, Section 6.3.3. An application of this very general inversion to deuteron scattering from ${}^4\text{He}$ at low energies is presented in Ref. [45].

6.3.7 Further extensions?

Spin-1 inversion, just discussed, is a restricted form of coupled channel inversion in which a non-diagonal S-matrix is inverted to give a non-diagonal potential of a very

specific form. Just how far the IP concept can be pushed to provide a more general form of coupled channel inversion is a challenge for the future. The difficulty of providing experimental data of sufficient breadth and precision might restrict the application of coupled channel inversion to non-diagonal S-matrices that have been calculated from theory.

7 Applications of inversion in nuclear scattering

There are a variety of good reasons for determining local potentials that reproduce given sets of S-matrices or phase shifts, and some of these will emerge from the account given here of the various applications of inversion. We cannot give an exhaustive list of possible applications, and readers may well be inspired to find new ones.

7.1 Nucleon-nucleon and similar interactions

Unlike nucleon-nucleus potentials, nucleon-nucleon interactions are not generally considered to be explicitly energy-dependent and are therefore a candidate for the application of fixed- l phase-shift-to-potential inversion. This would exploit the comprehensive phase-shift analyses covering a wide energy range. In fact ‘fixed- l ’ is a misleading term since the tensor interaction mixes l channels for given conserved total angular momentum and parity. The l -dependence of the potential and the small number of active partial waves effectively rule out fixed- E inversion.

The challenge of applying Gel’fand-Levitan-Marchenko methods, generalized to allow for coupled channels inversion to determine the non-diagonal tensor interaction, was successfully taken up by von Geramb and collaborators, see their articles [5].

7.2 ‘Two-step’ nuclear elastic scattering phenomenology.

It is often desirable to fit elastic scattering observables with potential models that have as few as possible *a priori* assumptions (or prejudices) concerning their nature. To achieve this, a number of ‘model independent’ fitting algorithms to determine optical model (OM) potentials, typically based on sums of gaussian or spline functions, have been developed. These often allow point-by-point uncertainties to be assigned to the potential.

The possibility of inversion affords an alternative approach and arguments in support of this approach have been made [14, 46, 47, 48, 49]. The idea is to first determine S_l or S_{lj} by fitting the elastic scattering observables (angular distributions (ADs) analyzing powers (APs) etc). These S_l or S_{lj} can then be inverted in a subsequent step. We refer to this overall procedure as ‘two-step nuclear (elastic scattering) phenomenology’. We here distinguish two classes of two-step phenomenology: (1) determination of the

S-matrix at one or a few discrete energies, and, (2) the fitting (usually for few-nucleon target nuclei) of functional forms $S_l(E)$ or $S_{lj}(E)$, over a fairly continuous range of energies, by means of an R-matrix or effective range procedure. We discuss these two cases separately, and show that they unambiguously reveal the parity dependence of many light-ion nucleus-nucleus interactions.

7.2.1 Discrete-energy OM fitting by inversion

When elastic scattering data, particularly data of typical precision and angular range, are fitted by searching on S-matrix elements, various profoundly different solutions may be found. Each S-matrix solution will lead, by inversion, to a different potential. In fact, the experience of carrying out such searches is very revealing about the under-determination of the potential by the data. Even so-called ‘model-independent’ OM fits often implicitly embody prior information that appears to ameliorate, but not completely remove, this problem. S-matrix searches at single energies, therefore, should be constrained with prior information; this is possible. We note a study of the effects of systematic errors in the analysis of nuclear scattering data [29].

There are various ways of incorporating prior information. One can start with an analytic form for S_l (e.g. McIntyre, Wang and Becker [50] (MWB)) and search on the parameters [46, 47], or one can start from an MWB or similar analytic form $S(l)$ and search on an additive component $\Delta S(l)$ in a way that preserves unitarity. Generally, it is important to regulate or constrain the search in appropriate ways. Useful, physically motivated, starting functions $S(l)$ for the S-matrix search are the S-matrices calculated by a Glauber model or by an existing phenomenological fit, see [48].

In Ref. [46], the elastic scattering of ^{16}O on ^{12}C at 608 MeV was analyzed by first determining S_l by fitting the elastic scattering data by means of an additive correction to the S_l of the McIntyre, Wang and Becker (MWB) [50] parameterized form. The correction was a searchable spline function of l . The angular distribution was first approximately fitted with a five parameter MWB form S_l ; the subsequent fitted spline function addition led to a threefold reduction in χ^2/N . The resulting corrected MWB-derived potential revealed a very different degree of surface transparency compared to the uncorrected MWB potential. At this energy the IP inversion is precise and stable, leading to effectively identical potentials independently of whether the iterative inversion of S_l started from a *zero* potential, $V(r) = 0$, or a Woods-Saxon potential in the neighbourhood of the expected result.

Ref. [47] discusses in depth the application of two-step phenomenology in an account of its application to $^{12}\text{C} + ^{12}\text{C}$ elastic scattering at 9 energies from 140 to 2400 MeV. This paper, in effect, presents a critique of standard OM phenomenology, with a discussion of the advantages (including computational efficiency) of two-step phenomenology and the means of implementing it. The strategy for avoiding spurious solutions was discussed, including the application of continuity-with-energy to the solutions, which, in this case, revealed apparent serious shortcomings of the data at one specific energy. This study also revealed weaknesses in the conventional Woods-Saxon phenomenology across the

whole energy range. At these energies, there is no problem in inverting S_l for just even values of l , as arises with the scattering of identical bosons.

An analysis of $^{16}\text{O} + ^{16}\text{O}$ elastic scattering was carried out [48], fitting wide angular range and precise data at a single energy, 350 MeV. Alternative solutions for S_l starting from the S-matrices calculated by a Glauber model or an existing phenomenological fit, led to very precise fits for $\sigma(\theta)/\sigma_{\text{Ruth}}(\theta)$ spanning nearly 5 orders of magnitude, leading to very similar potentials for $r \geq 4$ fm. There is little sensitivity at radii less than that, contradicting claims for repulsive effects at a small radius. Brandan and Satchler [26], their Section 9.2, evaluate the resulting potentials [48] in comparison with potentials obtained using alternative procedures. We note in passing that the simplified Glauber model gave $|S_l|$, but not $\arg S_l$, in close agreement with values from the final fitted S_l .

In Ref. [49] two-step inversion is applied to the elastic scattering of ^{11}Li from ^{28}Si (at 319 MeV) and from ^{12}C (at 637 MeV). This work revealed the profound ambiguities in S_l , and thus in $V(r)$, that occur when fitting data of limited range and precision. These ambiguities are more extreme than those that are found with conventional OM fitting. Good fits to the data were easily (too easily?) obtained in which the large- r tail was surprisingly extended, although this possibly results from contamination of the forward angle elastic AD data with inelastic scattering. Realizing the full potential of the two-step method for analyzing the elastic scattering of halo nuclei awaits the advent of sufficiently high quality data.

Inversion can, of course, be applied to an independently fitted published S-matrix. In Ref. [36] IP inversion of S_{lj} for $p + ^4\text{He}$ elastic scattering was carried out for S_{lj} that had been fitted to angular distribution data at a single energy, 64.9 MeV. The resulting potential was not of Woods-Saxon-like form, but exhibited long-wavelength oscillations. The reason for these became evident later with subsequent multi-energy inversions, see Section 7.2.2. Ref. [36] also presented real and imaginary, scalar and vector, Dirac equivalent to the Schrödinger potentials. This demonstrated that, with certain limitations, Dirac equation $S \rightarrow V$ inversion is possible by way of Schrödinger equation equivalence.

7.2.2 Inverting S-matrices from R-matrix and effective range fits

Mixed case and energy-dependent inversion make possible an alternative form of two-step inversion. The first step now takes the form of an R-matrix or effective range fit to elastic scattering data over a possibly quite wide energy range, which might include shape resonances. The result is an analytic form of S-matrix in which the (typically) small number of active partial waves is compensated for by the fact that $S_{lj}(E)$ exists for a substantial range of energy E . Satisfactory inversion becomes possible even in cases where the energy range is much less than is required for fixed- l inversion and, also, the number of partial waves is much fewer than what is required for fixed- E inversion. Inversion of this kind becomes particularly interesting when the results can be compared with the potentials derived from the inversion of $S_{lj}(E)$ from RGM or similar theories for the same few-nucleon systems; we discuss the specific case of nucleon

scattering from ${}^4\text{He}$ in Section 7.4.2.

In Section 7.2.1 it was mentioned that single-energy two-step inversion for $p + {}^4\text{He}$ led to wavy potentials. Multi-energy inversion provides an explanation and points to a significant property of nucleus-nucleus interactions between few-nucleon nuclei. Ref. [34] applied mixed case inversion to S_{lj} for $p + {}^4\text{He}$ that had been fitted, using R-matrix and effective range expansions, to experimental data at several discrete energies. As a result, the following alternative emerged: *either* (i) the potential is wavy (as it was for 64.9 MeV), *or*, (ii) there is a smooth but parity-dependent potential. The parity dependence is such that the odd-parity potential has a substantially greater range and volume integral than the even-parity potential. We shall see in Section 7.4.2 that exactly this form of parity dependence emerges from the inversion S_{lj} derived from theories that include exchange processes (excluding knock-on, Fock term, exchange). Ref. [34] found that the $p + {}^4\text{He}$ and $n + {}^4\text{He}$ *nuclear* interactions were essentially identical in the odd-parity channels (as required by charge symmetry) but differed somewhat for $r \leq 2$ for the even-parity channels.

In Ref. [34], only energies below the inelastic threshold were involved so the potentials were real, unlike those of Ref. [36] or those found in Ref. [41] in which energy dependent IP inversion was introduced. Potentials for $p + {}^4\text{He}$ scattering were found by inverting $S_{lj}(E)$ from various R-matrix and effective range fits to experimental data from zero energy to about 65 MeV. The resulting potentials fitted the shape resonances at energies below the inelastic threshold, and became complex above the threshold, reproducing the data reasonably well up to 65 MeV. The opening of a relatively small number of specific channels at various energies above the threshold revealed the limits of potential models in which the potential varies smoothly with energy. Such a model evidently requires either the ‘many open channel’ condition of the standard optical model, or, the ‘zero open channel’ situation that holds below threshold; neither is true between the threshold and 65 MeV for $p + {}^4\text{He}$. Refs. [34, 41] together demonstrated, on an empirical basis, that the real (and imaginary above threshold) central as well as spin-orbit components of the nucleon- ${}^4\text{He}$ potential are parity dependent; they also established the practicality of energy-dependent inversion. The key finding: over the whole energy range considered, the parity dependence of the real central part is such that the odd-parity component has both a longer range and a greater volume integral than the even-parity term. Thus, potentials that have a factor $(1 + \alpha(-1)^l)$ multiplying a single radial form (as have been applied in optical model fits) are too restrictive.

Parity dependence extends beyond 5-nucleon systems: IP inversion of S_l that had been fitted [51] to ${}^4\text{He} + {}^{12}\text{C}$ elastic scattering data yielded [52] a strongly parity-dependent potential that reproduced the scattering data very well, including the shape resonances. The potential differed in the surface region from previous phenomenological potentials that had been found in a conventional way; this is of possible astrophysical significance.

The ${}^3\text{He} + {}^4\text{He}$ interaction, which is also of astrophysical importance, was shown to be strongly parity-dependent by Cooper in Ref. [81] in which both empirical and theoretical S_{lj} were inverted. The bound state energy was also included as input to an IP inversion for the first time, contributing to the determination of the interaction.

7.3 Data-to-potential direct inversion

It is possible to combine the determination and the IP inversion of the S-matrix into one algorithm, see Refs [53, 54, 55, 44, 45]. Scattering data that has been measured for multiple energies can be included, in this way implementing energy-dependent, direct data-to-potential inversion.

In Ref. [54], for protons scattering from ^{16}O at 7 energies from 27.3 MeV to 46.1 MeV, very precise wide angular range AD and AP data were fitted using energy-dependent direct data-to-potential inversion. Parity-dependent real and imaginary central potentials and complex Wigner spin-orbit potentials were determined. The even-parity and odd-parity central potentials were smooth and behaved in a regular way with energy. The odd-parity (Majorana) terms were very like those found from the inversion, described elsewhere, of S_{lj} derived from RGM calculations of protons scattering from ^{16}O . Fits of equal quality lacking parity dependence would certainly require wavy potentials. This work by Cooper might reasonably be described as state-of-the-art nucleon scattering phenomenology for a single nucleon-nucleus pair; it conclusively establishes the parity dependence of the interaction between a nucleon and ^{16}O . As a result, we conclude that the omission of parity dependence from tests of folding model theories, as applied to nuclei as light as ^{16}O , will lead to misleading results.

In Ref. [55] a parity-dependent potential, including spin-orbit terms, that gave a fair simultaneous fit to ADs and APs for $^6\text{Li} + ^4\text{He}$ scattering at 19.6, 27.7 and 37.5 MeV, was found. The Majorana terms were essential for the fit in this few-nucleon system. This appears to be something one must presume to be required in all few-nucleon system inversions, see Section 7.4.2.

In Ref. [44], which introduced coupled-channel IP inversion for the scattering of spin-1 projectiles leading to a T_R tensor interaction, data-to-potential inversion was carried out fitting multiple energy data including angular distributions, vector analyzing powers and the three tensor analyzing powers for deuterons scattering from ^4He . The energy range was from 8 to 13 MeV for 6 discrete energies; 4000 data were fitted with a potential that included parity-dependent central and tensor terms. A subsequent study [45] fitted a wider range of energies, including the three D-state resonances. Strong, complex parity-dependent T_R tensor interactions were revealed.

Direct data-to- V inversion of data covering a substantial range of energies, leading to a T_R tensor interaction, with all components parity-dependent (where required by the data) and energy dependent, represents the most complete implementation of the IP inversion procedure.

7.4 Determination of potentials from theory

The inversion of S-matrices calculated from a theory is not subject to the problems that may arise, e.g. from noise and experimental uncertainties, when the S-matrix determined from scattering data is inverted. Applications of the inversion of calculated

S-matrices include: (i) determining in a natural and efficient way, the local potential that is S-matrix equivalent to non-local or l -dependent potentials; (ii) deriving potentials that represent the scattering for those theories (see e.g. Section 7.4.2 and Section 7.4.3) that calculate scattering directly without the intermediary of potentials; (iii) providing arguably the best method of calculating the dynamic polarization potential (DPP) due to the coupling of inelastic or reaction channels to the elastic channel.

7.4.1 Dynamic polarization potentials (DPP) by CC-plus-inversion

The contribution of inelastic processes to nucleus-nucleus interactions is represented by the dynamic polarization potential (DPP) [26, 43] the non-local and l -dependent form of which (for inelastic scattering, at least) was derived by Feshbach [57]. This has been calculated with various approximations, see for example Refs. [58, 59], and leads to a highly non-local and l -dependent expression. Moreover, the inclusion of coupling to transfer channels with full finite range coupling and the inclusion of non-orthogonality terms has never been achieved in such calculations. Finally, the results are not easily related to phenomenology since it is necessary to establish local and l -independent equivalent potentials from such calculations and this requires the calculation of S_{lj} from non-local and l -dependent potentials.

There is an alternative procedure for calculating DPPs that can handle reaction channels, coupling of all orders and (in principle) exchange processes: ‘coupled channel plus inversion’. In this method, a potential is first found by inverting the elastic channel S-matrix from a coupled channel calculation. When the bare potential of the CC calculation is subtracted from this inverted potential, the resulting difference potential is a local and l -independent representation of the DPP arising from the channel coupling. A full discussion of this procedure, its advantages and limitations, and a comparison with other methods, is given in Ref. [60]. The method used in earlier attempts to extract the contribution of channel coupling to the optical potential (see Ref. [61] and references cited there) was to refit the CC angular distributions, but this is subject to the many limitations of optical model fitting.

Many processes that contribute to the DPP can be studied with CC-plus-inversion: coupling to inelastic channels, reaction channels, particle or cluster exchange and projectile breakup. IP inversion was introduced in a study [20] of the contribution of projectile breakup to the ${}^6\text{Li}$ -nucleus interaction. Many other studies of the DPP due to breakup of ${}^6\text{Li}$ have been made; Ref. [62] described generic properties of the DPP that were common to the breakup of deuterons and ${}^6\text{Li}$. A recent paper on ${}^6\text{Li}$ breakup, Ref. [9] which has many references, compared DPPs derived using S -matrix inversion and those from an l -weighted TELP, see Section 5, and found marked differences. The DPP due to the breakup of the halo nucleus ${}^6\text{He}$ has also been studied [63, 64, 65]: both the real and imaginary parts have remarkably long tails attributable to Coulomb breakup. The tail on the imaginary part is absorptive for $r \geq 13$ fm but emissive for smaller r ; the real part is attractive at large radii, but with a sharp change to repulsion at about 15 fm. Potentials are presented out to 60 fm. In Ref. [66] DPPs are compared

for the breakup of ${}^6\text{Li}$, ${}^7\text{Be}$ and ${}^8\text{B}$ scattering from ${}^{58}\text{Ni}$. Breakup in which the Coulomb interaction plays a much smaller role was evaluated for protons scattering from ${}^6\text{He}$ in Ref. [67].

A stimulus to the development of IP inversion was the discovery [68, 69, 70] that the coupling to deuteron channels appeared to have a large effect on proton scattering. This raised the question of what contribution this coupling makes to the nucleon OMP. An early application [71] of spin- $\frac{1}{2}$ [40] IP inversion presented the effects of finite range $p \rightleftharpoons d$ coupling on the real and imaginary, central and spin-orbit terms of the $p + {}^{40}\text{Ca}$ OMP at 30.3 MeV. The effect of finite-range coupling (not included in the earlier work) on the DPP was shown and the individual and total contributions of lumped $\frac{5}{2}+$, $\frac{1}{2}+$ and $\frac{3}{2}+$ pickup states was presented. Later studies of the contribution to the proton OMP of $p \rightleftharpoons d$ coupling, including non-orthogonality terms previously omitted, are referenced in Ref. [60]; these include cases of proton scattering from halo nuclei, most recently [72, 73].

In Ref. [74] it was found that the coupling mass-three channels had a major effect of deuteron elastic scattering; inversion revealed large repulsive DPPs. Subsequent development of CRC codes permit the inclusion of non-orthogonality corrections and finite range interactions, not included in Ref. [74], and, at the same time, spin-1 inversion leading to the T_R had been developed. These advances were exploited in Ref. [75] which determined the real and imaginary, central, spin-orbit and tensor DPPs generated by coupling to mass-3 pickup channels for 52 MeV deuterons scattering from ${}^{40}\text{Ca}$. The volume integral of the real, central DPP is much smaller than before [74], but the magnitude point-by-point is not small, reflecting the wavy character of both real and imaginary components. This is indicative of l -dependence, Sect. 7.4.4, and could not have been picked up by refitting the elastic scattering angular distributions from CC calculations, as in Ref. [61].

J-weighted inversion. Inversion has not been developed for spin > 1 and Ref.[75] introduced and tested a means of achieving inversion leading to a meaningful central potential for projectiles with large spin. The idea is to define a J-weighted S-matrix:

$$\bar{S}_l = \frac{\sum_J (2J+1) S_u^J}{\sum_J (2J+1)} \quad (14)$$

that could be inverted in the usual way. It was found that for the spin-1 case studied in Ref.[75], the imaginary part of the J-weighted DPP was close to the imaginary part of the central DPP from the complete inversion. The real part was qualitatively reproduced. The J-weighted procedure was later applied to the breakup of ${}^6\text{Li}$, ${}^7\text{Be}$ and ${}^8\text{B}$ scattering from ${}^{58}\text{Ni}$ mentioned above, Ref. [66].

7.4.2 RGM, GCM and other few-body cases; exchange contributions

Inversion can play a particular role in supporting our understanding of the scattering of few-nucleon systems. Even a scattering system as simple as a nucleon plus ${}^4\text{He}$ becomes very complicated if all reaction channels, realistic NN interactions and a full

account of exchange processes are to be included. At least until recently, the standard methods of calculating scattering observables would be the application of resonating group methods (RGM) or generator coordinate methods (GCM) [76, 77]. Even for systems of 4 or 5 nucleons, greatly simplified nucleon-nucleon interactions, generally omitting tensor terms, are employed. How are such theories to be tested in view of the inevitably approximate fits to data? Inversion provides a partial solution. In Section 7.2.2 we described the inversion of S-Matrices that had been fitted to experimental observables and noted that definite qualitative features of the potentials emerged, e.g. parity-dependent potentials with specific differences between the strength and range of the even-parity and odd-parity terms. Such features are a consequence of the various exchange terms that are precisely those aspects of the scattering process that RGM and GCM calculations treat correctly. The inversions to be described support very well the general conclusions of RGM and GCM calculations. In particular, it becomes possible to study the contributions of the many different exchange processes, apart from knock-on exchange (c.f. Section 7.4.4), that are included in RGM-GCM calculations, including those that are responsible for parity dependence. LeMere *et al* [78, 79] discussed in qualitative terms the nature of the contributions specific exchange terms would make. Moreover, Baye [80] has made predictions concerning the way the effect of exchange processes, including those leading to parity dependence, depends upon the masses of the interacting nuclei. It is important to check such things, particularly since, as we have already seen, parity dependence is substantial for neutrons or protons scattering on ${}^4\text{He}$ and apparently important for $p + {}^{16}\text{O}$. Is it still important for $n + {}^{40}\text{Ca}$? Baye's work would suggest not, but we need to know since its presence or absence would make a difference in precise checks of folding model theory for $n + {}^{40}\text{Ca}$.

The $p + {}^4\text{He}$ case is of particular interest for two reasons: it is less intractable than most, and clear parity dependence has been found from the analysis of experimental data, as reported in Section 7.2.2. RGM calculations of S_{lj} for $p + {}^4\text{He}$ scattering, in some of which $d + {}^3\text{He}$ configurations were included, were inverted using mixed case inversion. The calculations were below threshold so the potentials were real. The inversion confirmed the key result from the inversion of empirical R-matrix phase shifts, that the potential shows strong parity dependence such that the odd-parity potential is of considerably greater range and volume integral than the even-parity term. More elaborate $p + {}^4\text{He}$ RGM calculations extending above the inelastic threshold with coupling to the $d + {}^3\text{He}$ channels were presented in Ref. [82]. The contribution of the breakup of the deuteron in the $d + {}^3\text{He}$ channels was studied. It was found that the reaction channels increased the Majorana term in the proton potential. In this work, the contribution of the $p + {}^4\text{He}$ channel to the $d + {}^3\text{He}$ interaction was determined for $S = \frac{1}{2}$ and $S = \frac{3}{2}$ channel spins.

A challenge for energy-dependent IP inversion was the inversion [83] of S_{lj} from 0 to 25 MeV for $p + {}^6\text{He}$, calculated [84] using RGM. No reaction channels were included, but parity dependence was allowed for in the inversion. Spin-orbit terms were also included. Two specific questions were posed: (i) is the energy dependence of the real, central Wigner term consistent with the energy dependence of the global optical potential? and, (ii) is there a Majorana term that is less than that for $p + {}^4\text{He}$ in a way that is consistent with the predictions of Baye [80]? Inversion yielded a single parity-

dependent potential with an energy-dependent form that fitted $S_{lj}(E)$ over essentially the whole energy range, and verified both points (i) and (ii). Point (i) is interesting since it shows that the global OM energy dependence extends down to mass 6. Since there were no reaction channels in the RGM, it suggests that, as widely assumed but sometimes doubted, the bulk of the energy dependence of the nucleon-nucleus interaction is, indeed, a product of knock-on exchange.

In Ref. [85], RGM S-matrices for the following cases were inverted and the potentials evaluated: $p + \alpha$, $n + \alpha$, $p + {}^3\text{He}$, $n + {}^3\text{H}$, $n + {}^6\text{Li}$, $n + {}^{16}\text{O}$ and $n + {}^{40}\text{Ca}$. For the cases where the channel spin exceeded $\frac{1}{2}$, potentials (without spin-orbit components) were determined separately for each value of the channel spin. The Majorana terms were generally large, and in cases where there were two values of channel spin the Majorana term was quite different for each value, usually in a way for which plausible physical reasons exist, in line with what had been suggested [78, 79]. This work verified the parity dependence of the $p + {}^{16}\text{O}$ interaction, and also found very little parity dependence for $n + {}^{40}\text{Ca}$, as long as the $l = 0$ partial wave was excluded. Apart from this last proviso, this is in close accord with the predictions of Baye [80].

The contributions, together and separately, of specific exchange terms for ${}^4\text{He} + {}^{16}\text{O}$, ${}^3\text{He} + {}^4\text{He}$ and ${}^3\text{H} + {}^4\text{He}$, were studied by inversion in Ref. [86]. Of the various conclusions that were drawn, we mention just one: when a parity-*independent* purely phenomenological imaginary potential is included in the RGM calculation, in order to enable a more reasonable comparison with experiment, the imaginary part of the inverted potential is parity dependent. This might be due to the fact that the real potentials for each parity had differing degrees of non-locality and hence differing Perey effects. Perey effects are discussed in Section 7.4.4 below where the effect of non-locality on the inverted imaginary potential is noted.

S-matrices, S_l , from RGM calculations [87] for ${}^{16}\text{O} + {}^{16}\text{O}$ elastic scattering, for seven energies from 30 MeV to 500 MeV, were inverted [88] using IP inversion. Since the direct (non-exchange) potential was fixed, this provided an energy-dependent local equivalent of the interaction generated by the exchange term. The authors [87] had determined $V(r)$ from the the RGM S_l using l -weighted WKB inversion, so the IP inversions constituted a test of l -weighted WKB inversion; this appeared to work quite well for energies of 150 to 300 MeV, but was not accurate at the lower energies studied, 30 - 59 MeV. At 150 MeV and below, the exchange terms are quite strongly repulsive at the nuclear surface but attractive at the nuclear centre. These effects are much smaller at the higher energies, there being no surface repulsion at 500 MeV.

7.4.3 Potential representation of KMT and comparable theories

Other theoretical formulations of nuclear scattering also produce an S-matrix directly, without the intermediate stage of generating a potential. Such methods do not as yet give perfect fits to observables, so it is of interest to compare the local potential that would reproduce the S_{lj} of the formalism with the well-established phenomenological OMP. An example of such a theory is that due to Kerman, McManus and Thaler [89],

KMT. More recent developments of KMT theory have included multiple scattering terms and Pauli blocking effects. In Ref. [12] first and second order KMT potentials for nucleon- ^{16}O scattering are calculated at 100 and 200 MeV by applying IP inversion to the corresponding KMT S-matrix. The volume integrals and rms radii of the four components of the inverted potentials are given, facilitating a comparison with established global phenomenology as well as with the results of alternative theories based on local density approximation nuclear matter theory. Ref. [12] gives references to earlier KMT calculations by the first three authors and others.

7.4.4 Inverting S_{lj} from non-local and explicitly l -dependent potentials

Non-locality: IP inversion, whether energy dependent or not, is a convenient means of determining the local equivalent of a non-local potential. The non-locality in the nucleon-nucleus interaction that is due to inelastic processes is not well established, but the non-locality due to knock-on exchange (Fock term) is well known, and is the major source of the energy dependence of the local OMP. The energy independent Perey-Buck [90] non-local potential, which fits nucleon elastic scattering over a wide energy range, is thought to represent the exchange non-locality in a simple parameterized way. If the S_{lj} for a non-local potential can be calculated (this is straightforward), then IP inversion immediately yields the local-equivalent potential. If $S_{lj}(E)$ is calculated over a wide range of energies, then energy-dependent IP inversion immediately yields the energy dependence of the local equivalent potential. This has been done [91] for the Perey-Buck potential, leading to a calculated energy dependence that well matches the energy dependence of the empirical OMP. IP inversion was also applied to S_{lj} that had been calculated from a potential in which the real part was of Perey-Buck non-local form but in which the imaginary part was local. The resulting local potential had an imaginary term that was reduced compared to that included with the non-local real potential; the reduction factor was just the Perey [92] reduction factor of the wave function within the nucleus that is due to the non-locality of the real potential.

The S-matrix for a microscopic non-local calculation of neutron- ^{16}O scattering was inverted in Ref. [93]. It was found that the damping of the wave function within the nucleus, following from an exact microscopic treatment of exchange, matched very closely the damping associated with the phenomenological Perey-Buck potential: the original Perey effect [92]. This leaves surprisingly little room for damping from reaction and inelastic processes suggesting, maybe in line with Austern's picture [94], that the non-locality arising from channel coupling redistributes flux, but this occurs without a global reduction in the magnitude of the wave function.

l -dependence: IP inversion also provides a means of finding the l -independent equivalent to an explicitly l -dependent potential; this must exist. Parity dependence is not the only form of l -dependence that has been proposed, on various grounds, as a property of phenomenological OMPs (the Feshbach theoretical potential is l -dependent and also see Refs. [95, 96, 97]). As we have emphasized, the S-matrix S_l that is calculated from an l -dependent potential can be inverted to yield an l -independent equivalent. There is a motivation for doing so: if model-independent OM phenomenology pro-

duced a local potential of a form that was recognizably equivalent to an l -dependent potential, that might be regarded as evidence that there is ‘really’ l -dependence of that form. As an example, in Ref [13], S_l from l -dependent potentials that had been fitted to $^{16}\text{O} + ^{16}\text{O}$ scattering for energies from 30 MeV to 150 MeV were inverted. The real and imaginary l -dependencies were of quite different forms according to quite different physical motivations. The inverted potentials varied with energy in a systematic way. However, the imaginary part in particular was quite unlike any found with standard optical model phenomenology, casting some doubt on the particular l -dependence of the imaginary potential introduced by Chatwin *et al* [98].

In Ref. [99], local and l -independent DPPs arising from pickup coupling were found that had a significant emissive region at the nuclear centre. The possibility that this points to an l -dependent underlying DPP is supported by earlier [100] model calculations. In these model calculations, S_l calculated for an explicitly l -dependent local potential were inverted leading to an imaginary term with a strong emissive region at the nuclear centre. Note that such an emissive region does not imply unitarity breaking: an l -dependent potential can be devised for which $|S_l| \leq 1$ for all l , and for which the l -independent equivalent potential nevertheless has local emissive regions. Note that the emissive region reported in Ref. [99] was in the DPP, not the full potential.

In Section 7.2.2, it was reported that nucleon- ^4He scattering presented the following alternative: the potential exhibited either *waviness* or *parity dependence*. RGM calculations clearly imply a preference for parity dependence. Exchange processes that occur with much heavier scattering pairs of nuclei are also believed to lead to parity dependence. Michel and Reidemeister [101] presented strong evidence that the $^4\text{He} - ^{20}\text{Ne}$ interaction at 54.1 MeV contained a Majorana (i.e. $(-1)^l$) term. Cooper and Mackintosh [102] applied IP inversion to the S_l derived from the potential of Ref. [101] and found a parity-independent potential giving the same S_l . It had a substantial oscillatory feature, suggesting that the ‘waviness or parity-dependence’ alternative is a general feature. In this case, a quite small Majorana term led to quite a considerably wavy l -independent equivalent. This is not an argument against parity dependence, but, apart from showing the power of IP inversion, it also suggests that wavy potentials found in model independent OM fitting should be considered seriously as a clue to underlying parity dependence. Therefore, seeking perfect fits to elastic scattering data, even when wavy potentials result, should not be dismissed as ‘fitting elephants’ (see page 223 of Ref. [26]); all the information content of the experimental elastic scattering data can be given meaning — that is surely desirable.

8 Summary and outlook

For a wide range of energies and projectile-target combinations, $S \rightarrow V$ inversion, and also observable $\rightarrow V$ inversion, are straightforward, and would be routine if the possible applications were more widely appreciated. This review has concentrated on giving some account of the information concerning nuclear interactions that has been obtained, and left to other reviews the task of an exposition of the mathematical basis

of inverse scattering and the attendant formal problems.

We note here some present limitations to the application of inversion, with the hope that others might rise to the challenge of solving them:

Limitation 1: Spin. At present successful inversions are routine for the following cases: spin-zero on spin-zero, spin- $\frac{1}{2}$ on spin zero; spin-1 on spin-zero. Spin-1 on spin-zero is currently limited to determining the T_R interaction, the best established tensor interaction. This limitation on spin gets in the way of desired calculations mostly for the inversion of S-matrix elements from theory rather than for S-matrices extracted from experiments. For example, the calculation of the DPP for ${}^7\text{Li}$ or ${}^9\text{Be}$ breakup cannot exploit the full S-matrix from a CC calculation in which the spin of the projectile was treated properly, even for a spin-zero target. The multiplicity of possible tensor forms is discouraging, to say the least. However, a suitable weighted S-matrix, such as that defined in Eqn. 14, can determine at least the central potential implicit in the S-matrix output from a coupled channel calculation with particles of spin $> \frac{1}{2}$ in the elastic channel. This approach was used for deuterons in Ref. [75] and for ${}^6\text{Li}$ ($1+$), ${}^7\text{Be}$ ($\frac{3}{2}^-$) and ${}^8\text{B}$ (2^+) in Ref. [66], but no spin-orbit potentials could be extracted.

Useful inversions can be performed in cases where both projectiles have spin: in Ref. [85] separate potentials (without spin-orbit interaction) were derived for each of the two values of channel-spin for $p + {}^3\text{He}$ and $n + {}^6\text{Li}$ scattering (in both cases, the Majorana term was very different for each value of channel spin.) However, full inversion for higher spin remains a challenge.

Limitation 2: Coupled channel inversion. The full and practical solution to the problem

$$S_{\alpha\alpha'}^{J\pi} \rightarrow V_{\alpha\alpha'}(r) \quad (15)$$

remains elusive, though there have been proposed extensions of the NS method. The one fully successful coupled channel inversion procedure is the IP extension described above leading to the non-diagonal T_R interaction for spin-1 projectiles. This suggests that further extensions are possible, although the profusion of possible non-diagonal potentials is challenging. Coupled channel inversion might make it possible to answer questions such as: how do coupled reaction channels [103] modify the deformation parameters that emerge in the analysis of inelastic scattering from deformed nuclei?

Finally, a personal viewpoint: understanding the nucleon-nucleus interaction potential is of fundamental importance in nuclear physics. Much progress has been made in achieving a unified perspective for positive and negative energies. Nevertheless, some form of local density approximation is implicit in most studies and the extracted potentials are local and l -independent. The non-locality due to knock-on exchange is included in an approximate way, but the role of other forms of non-locality, which are known to be present, is obscure at best. Moreover, there is little understanding of how the interaction depends upon the particular last-occupied orbitals or collectivity of individual nuclei. Coupled channel-plus-inversion offers scope for understanding the processes which occur when a nucleon, and particles in channels coupled to the nucleon channels, interact with the curved nuclear surface, with its density gradients.

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